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# Analysis of Junction Conditions in Optimal Aeroassisted Orbital Plane Change

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**The atmospheric pass of the optimal heating-rate constrained aeroassisted orbital plane-change problem is considered. The problem formulation includes the thrust magnitude as a control variable. The junction conditions are the optimality conditions that must be satisfied by every pair of adjoining subarcs of the optimal trajectory. An analysis of these junction conditions shows that a number of prior assumptions on concatenating optimal subarcs must be revised. This is because the Hamiltonian of the problem is no longer regular due to the inclusion of thrust magnitude as a control variable. Necessary conditions for jump discontinuities in the angle of attack and thrust control are derived. We show that these necessary conditions permit a nontangential entry. Our conditions also show that a number of extremals hitherto not considered in the literature should be included in the construction of the switching structure. In addition, a number of new junction conditions are derived that may be used by numerical analysts to validate their codes. One interesting consequence of our analysis is a demonstration of the dual relationship between singular points and touch points.**

## I. Introduction

AEROASSISTED orbital transfer, or synergetic maneuvering, has been studied quite extensively for a very long time.[1] To circumvent some of the difficulties that arise in determining the optimal synergetic maneuver, many researchers solve the optimal control problem for some simplified modes of this maneuver.[2] The aeroglide and aerocruise maneuvers are two typical modes of the synergetic maneuver: The former employs no thrusting during the aerodynamic turn, whereas the latter accomplishes the maneuver by continuous thrusting. The aerodynamic heating rate has long been identified as one of the major constraining factors for the satisfactory performance of the synergetic maneuver. Consequently, it is customary to incorporate a heating-rate model as a constraint in solving the optimal control problem. Recently, Seywald[3] obtained variational solutions for the aeroglide maneuver and showed that nontrivial touch points were unlikely to be a part of the optimal solution. A critical part of his analysis was based on the junction conditions necessary for optimality. The junction conditions offer a key to how the various extremal arcs may be joined to form the optimal trajectory. Because it has long been known that thrusting inside the atmosphere has the potential to increase the synergetic efficiency, it is worthwhile to extend Seywald's results by including the thrust magnitude as a control variable. This inclusion dramatically increases the complexity of the problem because it allows discontinuous controls. As a result many of the stronger junction conditions applicable to the so-called regular Hamiltonian cannot be applied. In this paper we derive the set of junction conditions that do not require such regularity assumptions and explore some conditions related to the discontinuity of controls. Our analysis indicates that nontangential entries to the heating-rate boundary are possible, and these are characterized by continuous costates. When the costates are discontinuous, the entry is tangential and it allows jumps in the angle of attack as well as the thrust magnitude. These jump conditions may be written explicitly in terms of the states and control variables, thus allowing one to analyze these conditions without direct recourse to optimal control theory.

## II. Problem Formulation

The basic synergetic maneuver begins with a retrorocket burn that lowers the perigee well into the atmosphere. During the atmospheric pass, the vehicle is banked so that the lift vector is in a lateral direction, thus producing an aerodynamic turn. The aerodynamic turn may be powered or unpowered. Following the aerodynamic turn, the vehicle is reboosted to the desired orbital altitude with an additional rocket burn to circularize the orbit.

Although the purpose of the synergetic maneuver is to minimize propellant consumption, we will assume a more general measure of trajectory performance given by the Mayer index

$$J[\boldsymbol{u}] = F(\boldsymbol{x}_f) \tag{1}$$

where $\boldsymbol{x}$ is the seven-dimensional state vector $\boldsymbol{x} = [r, \theta, \varphi, V, \psi, \gamma, m]^T$ that consists of the radial position, longitude, latitude, speed, heading angle, flight-path angle, and mass of the spacecraft, respectively. The subscript $f$ denotes the final condition. The control vector $\boldsymbol{u} = [\delta, \alpha, T]^T$ consists of the bank angle, the angle of attack, and thrust magnitude, respectively. We will assume that the thrust magnitude is constrained by

$$0 \le T \le T_{\max} \tag{2}$$

while the angle of attack and the bank angle are unconstrained. Not imposing a constraint on $\alpha$ is a reasonably serious issue because all vehicles stall at sufficiently high angles of attack. The two constraints suggested by Eq. (2), that is, $T \ge 0$ and $T \le T_{\max}$ may be combined as a single constraint

$$T^2 - T T_{\max} \le 0 \tag{3}$$

This allows us to use a single Lagrange multiplier for the thrust constraint.

Nearly all of the interesting problems of this maneuver occur in the atmosphere. In limiting the scope of this paper, we restrict our attention to the atmospheric pass. A significant body of prior work contains the assumption of an unpowered turn.[1] As we do not make this assumption, the equations of motion during the atmospheric pass are given in Ref. 4,

$$\dot{r} = V \sin\gamma \tag{4a}$$

$$\dot{\theta} = \frac{V\cos\gamma\cos\psi}{r\cos\varphi} \tag{4b}$$

$$\dot{\varphi} = \frac{V\cos\gamma\sin\psi}{r} \tag{4c}$$

$$\dot{V} = a_s - g\sin\gamma \tag{4d}$$

$$\dot{\psi} = \frac{a_w}{V\cos\gamma} - \frac{V}{r}\cos\gamma\cos\psi\tan\varphi \tag{4e}$$

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$$\dot{\gamma} = \frac{a_n}{V} - \left(g - \frac{V^2}{r}\right)\frac{\cos\gamma}{V} \tag{4f}$$

$$\dot{m} = -\frac{T}{v_e} \tag{4g}$$

where $g = \mu/r^2$ is the inverse-square gravitational acceleration, $v_e$ is the rocket exhaust speed, and $a_s$, $a_n$, and $a_w$ are, respectively, the nongravitational perturbing accelerations on the vehicle in the tangential, normal, and binormal directions given, respectively, by

$$a_s = (T\cos\alpha - D)/m \tag{5a}$$

$$a_n = (L + T\sin\alpha)\cos\delta/m \tag{5b}$$

$$a_w = (L + T\sin\alpha)\sin\delta/m \tag{5c}$$

In the preceding equations, $L$ and $D$ are the aerodynamic lift and drag, respectively, given by

$$L = qAC_L \tag{6a}$$

$$D = qAC_D = qA(C_{D_0} + kC_L^2) \tag{6b}$$

where $q$ is the dynamic pressure given by

$$q = \rho(r)V^2/2 \tag{6c}$$

and $\rho$, $A$, $C_L$, and $C_D$ are, respectively, the atmospheric density, the vehicle reference area, lift coefficient, and drag coefficient. Note that the thrust vector is assumed to be fixed along the reference body axis of the vehicle from which $\alpha$ is measured. It is important to emphasize that an independent thrust vector control, that is, a gimbaled system, is not considered here. As suggested by Eq. (6b), the drag coefficient $C_D$ is assumed to have a standard quadratic drag polar where $C_{D_0}$ and $k$ are, respectively, the zero-lift drag coefficient and the induced drag parameter. Generally speaking, the atmospheric density is functionally dependent on position and time: $\rho = \rho(r, \theta, \varphi, t)$; however, for the purpose of generating additional integrals of motion, we assume it to be time invariant, spherically symmetric, and expressible by a local exponential

$$\rho(r) = \rho_0 \exp[-\beta(r - r_0)] \tag{7}$$

In choosing the spherical coordinates for the equation of motion, we will ignore the singularities arising due to the choice of the model; hence, we will assume [see Eq. (4)]

$$r \neq 0, \qquad V \neq 0, \qquad m \neq 0, \qquad \cos\varphi \neq 0, \qquad \cos\gamma \neq 0 \tag{8}$$

In addition, we will assume that

$$L + T\sin\alpha > 0 \tag{9a}$$

$$L_\alpha + T\cos\alpha > 0 \tag{9b}$$

$$D_\alpha + T\sin\alpha > 0 \tag{9c}$$

where $L_\alpha$ and $D_\alpha$ are the partial derivatives of lift and drag with respect to the angle of attack. All three of these assumptions are reasonable, as they are valid for a large class of vehicles. The first assumption [Eq. (9a)] is automatically satisfied if $T = 0$ because $L > 0$. If $T \neq 0$, then a sufficient condition for this assumption to be valid is that the optimal angle of attack be nonnegative. For Eqs. (9b) and (9c), it is assumed that the $L_\alpha > 0$ and $D_\alpha > 0$ if $T = 0$ and that if $T \neq 0$, then $0 < \alpha < 90$ deg is a sufficient condition for these equations to be valid.

During the atmospheric pass, the vehicle may be subjected to very high thermal loads. Employing a thermal protection system to withstand large heating rates may be counter-productive to minimizing fuel consumption. Consequently, it is important to consider a heating-rate constraint. The aerodynamic heating rate on the spacecraft is constrained to lie in an allowable region, $h(r, V) \leq 0$, specified by[2]

$$h(r, V) = Q(r, V) - Q_{\max} = C\rho^N(r)V^M - Q_{\max} \leq 0 \tag{10}$$

where $C$, $N$, and $M$ are constants and $M$ is positive. This is a first-order state constraint and may be viewed as an indirect constraint on the controls because over the boundary arc $[h(r, V) = 0]$.

## III. Issues in the Formulation of the Problem

The main problem is to determine the optimal open-loop control program $\boldsymbol{u}^*(t)$ that maneuvers the rocket vehicle from a given initial condition, $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, to a target manifold, $\boldsymbol{y}(\boldsymbol{x}_f, t_f) = \boldsymbol{0}$, $\boldsymbol{y} \in \Re^s$, $s \leq 7$, while minimizing the Mayer performance index given by Eq. (1). Broadly speaking, we can break up the trajectory into two types of segments: one type, called exterior arc, where the heating rate is below the maximum ($h < 0$) and the other type, called boundary arc, where the heating rate is at the maximum value ($h = 0$). Where two different arc types meet, we have a junction point. The junction points can be classified as either entry points or exit points. If the entry and exit points degenerate to a single point, it is called a touch point (or, sometimes, a contact point). When an exterior arc meets a boundary arc, that is, a junction point, certain conditions called jump conditions have to be met for the optimality of the full trajectory. These are important necessary conditions because it narrows the search space for the optimal trajectory.

As the thrust magnitude (a control variable) appears linearly in the state equations, it allows the possibility of singular thrust arcs: a special type of thrust modulation that extremizes the performance index. Thrust modulation, other than singular will, by definition, yield nonextremals. Despite this, the aerocruise maneuver has enjoyed extensive acceptance as an optimal maneuver without any association with it being singular. A full discussion of singular arcs is beyond the scope of this paper, but some discussion of this may be found in Ref. 5. Consequently, we limit our analysis to nonsingular (or regular) arcs.

## IV. Extremal Arcs

The extremal arcs are those subarcs that satisfy the minimum principle. This principle requires that the Hamiltonian be minimized over all of the allowable controls. The Hamiltonian is given by

$$\begin{aligned} H = {} & \lambda_r V\sin\gamma + \lambda_\theta \frac{V\cos\gamma\cos\psi}{r\cos\varphi} + \lambda_\varphi \frac{V\cos\gamma\sin\psi}{r} - \lambda_V g\sin\gamma \\ & - \lambda_\psi \frac{V}{r}\cos\gamma\cos\psi\tan\varphi - \lambda_\gamma\left(g - \frac{V^2}{r}\right)\frac{\cos\gamma}{V} \\ & + \lambda_V \frac{T\cos\alpha - D}{m} + \lambda_\psi \frac{(L + T\sin\alpha)\sin\delta}{mV\cos\gamma} \\ & + \lambda_\gamma \frac{(L + T\sin\alpha)\cos\delta}{mV} - \lambda_m \frac{T}{v_e} \end{aligned} \tag{11}$$

where the $\lambda_r$, $\lambda_\theta$, $\lambda_\varphi$, $\lambda_V$, $\lambda_\psi$, $\lambda_\gamma$, and $\lambda_m$ are the components of the costate $\boldsymbol{\lambda}$. For the exterior arcs, the only constraint is on the thrust magnitude [cf. Eq. (3)] whereas the boundary arc restricts the control space further. A widely used method for handling the state constraint is to convert it to an equality constraint on the control and then minimize the Hamiltonian (in the subspace). This indirect adjoining of the constraints to the Hamiltonian yields an indirect Lagrangian (also called the augmented Hamiltonian) and is sometimes referred to as the $P$ form. This method is quite useful in numerical applications as it yields an explicit expression for the Lagrange multiplier associated with the state constraint.[6–8] However, from a theoretical point of view, a simpler way to handle all of the constraints is to directly adjoin them to the Hamiltonian and then employ variational techniques. The resulting necessary conditions are equivalent to investigating the Karush–Kuhn–Tucker conditions on minimizing the Hamiltonian with the state constraint considered as a direct constraint on the Hamiltonian.[6–8] This is the $D$ form of the Lagrangian and is given by

$$\Lambda = H + \nu h + \mu(T^2 - TT_{\max}) \tag{12}$$

where $\nu$ and $\mu$ are the Lagrange multipliers associated with the nonlinear problem of minimizing $H$ that satisfy the complementary slackness conditions,

$$\nu h = 0 \qquad \nu \geq 0 \tag{13a}$$

$$\mu(T^2 - TT_{\max}) = 0 \qquad \mu \geq 0 \tag{13b}$$

In this framework, the costates must satisfy the differential equations

$$\dot{\lambda} = -\frac{\partial \Lambda}{\partial \boldsymbol{x}} = -\frac{\partial H}{\partial \boldsymbol{x}} - \nu \frac{\partial h}{\partial \boldsymbol{x}} \tag{14}$$

The optimality condition requires that $\partial \Lambda / \partial \boldsymbol{u} = \boldsymbol{0}$, which yields

$$\frac{\partial \Lambda}{\partial \delta} = \frac{\partial H}{\partial \delta} = \frac{L + T\sin\alpha}{mV}\left(\lambda_\psi \frac{\cos\delta}{\cos\gamma} - \lambda_\gamma \sin\delta\right) = 0 \tag{15a}$$

$$\frac{\partial \Lambda}{\partial \alpha} = \frac{\partial H}{\partial \alpha} = -\lambda_V \frac{T\sin\alpha + D_\alpha}{m} + \left(\frac{L_\alpha + T\cos\alpha}{mV}\right)\left(\lambda_\psi \frac{\sin\delta}{\cos\gamma} + \lambda_\gamma \cos\delta\right) = 0 \tag{15b}$$

$$\frac{\partial \Lambda}{\partial T} = \lambda_V \frac{\cos\alpha}{m} + \lambda_\psi \frac{\sin\alpha\sin\delta}{mV\cos\gamma} + \lambda_\gamma \frac{\sin\alpha\cos\alpha}{mV} - \lambda_m \frac{1}{v_e} + \mu(2T - T_{\max}) = 0 \tag{15c}$$

Note that Eqs. (15) are valid for the entire trajectory even though the Lagrange multiplier associated with the heating rate $\nu$ does not appear explicitly in these equations. This is a theoretical advantage as these equations indeed provide the totality of extremals. It is quite important to exercise care in solving Eqs. (15) due to the possibility of some of the Lagrange multipliers vanishing. Deferring a discussion on solving these equations, we first note that some insight on the nature of the extremals arcs can be obtained by considering the additional equations that arise when the trajectory lies on the heating-rate boundary.

If the trajectory lies on the heating-rate boundary, then the necessary and sufficient condition for it to remain on it is given by

$$\frac{\mathrm{d}^i}{\mathrm{d}t^i} h = 0, \qquad i = 0, 1, 2, \ldots \tag{16}$$

For $i = 1$, it is straightforward to show that

$$T\cos\alpha - D - m\sin\gamma[g + (\beta N/M)V^2] = 0 \tag{17}$$

which indicates that the heating rate is a first-order state constraint.

The second-order necessary conditions provide significant insight on the totality of extremals arcs. The Legendre–Clebsch condition requires that $\partial^2\Lambda/\partial\boldsymbol{u}^2$ (evaluated at the optimal point) be positive semidefinite. It is apparent that $\partial^2\Lambda/\partial\boldsymbol{u}^2$ may not be of full rank. Consequently, we can classify extremals as either regular [when $\det(\partial^2\Lambda/\partial\boldsymbol{u}^2) \neq 0$] or singular [when $\det(\partial^2\Lambda/\partial\boldsymbol{u}^2) = 0$].

**Regular Extremals**

The Legendre–Clebsch condition on the bank angle yields

$$\frac{\partial^2 \Lambda}{\partial \delta^2} = \frac{\partial^2 H}{\partial \delta^2} = \frac{-(L + T\sin\alpha)}{mV}\left(\lambda_\psi \frac{\sin\delta}{\cos\gamma} + \lambda_\gamma \cos\delta\right) \geq 0 \tag{18}$$

With the aid of Eq. (9a), this simplifies to

$$\lambda_\psi(\sin\delta/\cos\gamma) + \lambda_\gamma \cos\delta \leq 0 \tag{19}$$

In any case, from Eq. (15a), the optimal bank angle profile is now given by

$$\tan\delta = \frac{\lambda_\psi}{\lambda_\gamma \cos\gamma} \tag{20}$$

which includes the possibility of $\lambda_\gamma = 0$, in which case $\delta = \pi/2$ if $\lambda_\psi < 0$ or $\delta = -\pi/2$ if $\lambda_\psi > 0$. When combined with Eqs. (9b), (9c), and (15b), Eq. (19) yields the powerful condition

$$\lambda_V \leq 0 \tag{21}$$

which generalizes the one obtained by Seywald[3] and Ross[5] in the sense that it remains valid over an arbitrary, nonzero thrust arc as well. This is especially significant because the inclusion of thrust components do not allow us to use the lift coefficient as the control variable (which greatly simplifies the task); rather, we have to work with the more primitive angle of attack. We will now use Eq. (21) to derive a second-order optimality condition as follows.

From Eq. (15b), we have

$$\frac{\partial^2 \Lambda}{\partial \alpha^2} = -\lambda_V \frac{T\cos\alpha + D_{\alpha\alpha}}{m} + \left(\frac{L_{\alpha\alpha} - T\sin\alpha}{mV}\right) \times \left(\lambda_\psi \frac{\sin\delta}{\cos\gamma} + \lambda_\gamma \cos\delta\right) \tag{22}$$

Using Eq. (15b) again to eliminate the last term in parenthesis of Eq. (22), we can write

$$\frac{\partial^2 \Lambda}{\partial \alpha^2} = -\frac{\lambda_V}{m}\left[T\cos\alpha + D_{\alpha\alpha} + (T\sin\alpha - L_{\alpha\alpha})\frac{T\sin\alpha + D_\alpha}{T\cos\alpha + L_\alpha}\right] \tag{23}$$

From Eqs. (9b) and (21), the second-order optimality condition on the angle of attack (and thrust) reduces to

$$(T\cos\alpha + L_\alpha)(T\cos\alpha + D_{\alpha\alpha}) + (T\sin\alpha + D_\alpha)(T\sin\alpha - L_{\alpha\alpha}) \geq 0 \tag{24}$$

which is quite a useful expression because it depends only on the control variables and the states. The optimal angle of attack follows from Eq. (15b). Substituting Eq. (20) into Eq. (15b) yields a pair of simpler equations given by

$$\frac{L_\alpha}{V\cos\delta} - \frac{\lambda_V D_\alpha}{\lambda_\gamma} = T\left(\frac{\lambda_V \sin\alpha}{\lambda_\gamma} - \frac{\cos\alpha}{V\cos\delta}\right) \qquad \text{if} \quad \lambda_\gamma \neq 0 \Leftrightarrow \delta \neq \pm\pi/2 \tag{25a}$$

$$\lambda_V D_\alpha + \frac{|\lambda_\psi| L_\alpha}{V\cos\gamma} = -T\left(\frac{|\lambda_\psi|\cos\alpha}{V\cos\gamma} + \lambda_V \sin\alpha\right) \qquad \text{if} \quad \lambda_\gamma = 0 \Leftrightarrow \delta = \pm\pi/2 \tag{25b}$$

If $T \neq 0$, neither of these equations is explicitly solvable for angle of attack. The special case of $T = 0$ (known as the aeroglide maneuver) has recently been discussed by Seywald.[3]

From Eq. (15c) it is clear that

$$\frac{\partial^2 \Lambda}{\partial T^2} = 2\mu \tag{26}$$

From the requirement of the strict form of the Legendre–Clebsch condition, it follows that $\mu > 0$. Combining this with the complementary slackness condition of Eq. (13b), it is clear that the thrust program is bang-bang, and Eq. (15c) simplifies to

$$T = \begin{cases} 0 & \text{whenever} \quad S > 0 \\ T_{\max} & \phantom{\text{whenever}} \quad S < 0 \end{cases} \tag{27}$$

where $S$ is the switching function defined by

$$S = \lambda_V \frac{\cos\alpha}{m} + \lambda_\psi \frac{\sin\alpha\cos\delta}{mV\cos\gamma} + \lambda_\gamma \frac{\sin\alpha\cos\delta}{mV} - \lambda_m \frac{1}{v_e} \tag{28}$$

Note that all of the preceding equations (in this subsection) hold for the entire trajectory, that is, whether or not it lies on the boundary arc. For the boundary arc, Eq. (17) must also hold. Combining this with Eq. (27) yields a state feedback form for the angle of attack given implicitly by

$$-D(\alpha_{ba}) - m\sin\gamma[g + (\beta N/M)V^2] = 0 \qquad \text{when} \qquad S > 0 \tag{29a}$$

$$T_{max}\cos\alpha_{ba} - D(\alpha_{ba}) - m\sin\gamma[g + (\beta N/M)V^2] = 0 \qquad \text{when} \quad S < 0 \tag{29b}$$

where $\alpha_{ba}$ is the angle of attack over the boundary arc. From Eq. (29a), it is clear that a boundary arc is possible (over a null thrust arc) only when $\gamma < 0$ (because the remaining terms are positive), or equivalently, only during the portion of the trajectory where the altitude is dropping. Intuitively, one must have a longer boundary arc to increase the change in inclination. This is possible by a powered aerodynamic turn as suggested by Eq. (29b). This powered turn is a maximum thrust maneuver, which has the advantage of performing a quick turn at the node.

**Singular Extremals**

When $\det(\partial^2\Lambda/\partial\boldsymbol{u}^2) = 0$, the extremal arc is said to be singular. It is totally singular if $\partial^2\Lambda/\partial\boldsymbol{u}^2 = \boldsymbol{0}$, otherwise it is partially singular. A full discussion of singular arcs is beyond the scope of this paper; however, for the purpose of completeness, it is worth elaborating on a few key issues. Our discussion will also be limited to the case of partially singular arcs, which is possible when $\mu = 0 \Leftrightarrow \partial^2\Lambda/\partial T^2 = 0$ [see Eq. (26)]. Thus, from Eq. (13b), we have $0 < T < T_{max}$. The singular thrust program is of first order and for the exterior arc, it can be obtained from $d^2(\partial\Lambda/\partial T)/dt^2 = 0$. Carrying out this operation and solving for the singular thrust is not a trivial task, and some very restricted cases are derived in Refs. 9 and 10. The singular thrust program given by Ref. 9 is valid only for a constant altitude profile, whereas the equations in Ref. 10 are valid for zero angle of attack. In general, neither the altitude is a constant nor is angle of attack zero over a singular thrust program. In fact, it is given by a very simple and elegant expression[5]

$$\tan\alpha = \frac{D_\alpha}{L_\alpha} = \frac{\partial C_D}{\partial C_L} \tag{30}$$

A simple check shows that this angle of attack is typically greater than that at maximum lift-to-drag ratio.

For a boundary arc, it is not necessary to repeatedly differentiate the switching function to determine an expression for the singular thrust. This exercise may be obviated inasmuch as we can solve for thrust from Eq. (17) and arrive at the nonlinear state feedback law,

$$T_{ba} = \frac{D + m\sin\gamma(g + \beta N V^2/M)}{\cos\alpha} \tag{31}$$

where the subscript ba is the boundary arc. Note that because $\det(\partial^2\Lambda/\partial\boldsymbol{u}^2) = 0$,

$$\frac{\partial^2\Lambda}{\partial\alpha\partial T} = \frac{\partial^2\Lambda}{\partial T\partial\alpha} = 0 \tag{32}$$

yields the same integrals as in Ref. 5. This is direct consequence of using the $D$ form of the Lagrangian. Hence, Eq. (30) is also valid over a boundary arc, and consequently, the same angle of attack holds over the singular thrust program whether or not it lies on the boundary.

## V. Junction Conditions

The optimal trajectory is formed by concatenating extremal subarcs described in the preceding sections. For nonlinear numerical programs such as the present one, there is no known theory to determine this switching structure. However, one can eliminate a few possibilities by investigating the junction conditions. The junction conditions are those conditions that must be satisfied at the point of entry or exit of the boundary arc (or also between regular and singular extremals). For any entry time $\tau$, the costate $\boldsymbol{\lambda}$ and the Hamiltonian $H$ may have a discontinuity given by the following jump conditions[6]:

$$\boldsymbol{\lambda}(\tau^+) = \boldsymbol{\lambda}(\tau^-) - \eta\frac{\partial h}{\partial \boldsymbol{x}} \tag{33}$$

$$H(\tau^+) = H(\tau^-) + \eta\frac{\partial h}{\partial t} \tag{34}$$

$$\eta \geq 0 \tag{35}$$

where $\tau^-$ and $\tau^+$ are evaluations of the functions at the left- and right-hand-side limits, respectively. The Hamiltonian is continuous across the junction points because the state inequality constraint does not explicitly depend upon the independent variable, time. Note that the signs on $\eta(\geq 0)$ in Eqs. (33) and (34) will be reversed if the maximum principle is employed. Conditions identical to Eqs. (33–35) hold for an exit time and will not be discussed any further. When the state constraint is of first order (as is the case here), then we have the following additional result.[6]

*Proposition 1:* The Lagrange multiplier $\eta = 0$ if either of the following conditions holds.

1) The control is continuous across the junction point, and the gradients of all of the active constraints are linearly independent.

2) The entry or exit is nontangential, that is, $dh(\tau^-)/dt > 0$ or $dh(\tau^+)/dt < 0$.

From condition 1, it is clear that we have

$$\eta\frac{dh(\tau^-)}{dt} = 0 \tag{36}$$

Because we also have $\eta\, dh(\tau^+)/dt = 0$, Eq. (36) may also be written in a more useful format as

$$\eta\left[\frac{dh(\tau^-)}{dt} - \frac{dh(\tau^+)}{dt}\right] = 0 \tag{37}$$

A number of additional junction theorems found in the literature[6,11] are mostly applicable to regular Hamiltonians, that is, Hamiltonians that yield unique maximizing controls. Other theorems found in the literature that do not require the assumption of regularity are valid only for a single scalar control.[11] Thus, a good many additional theorems found in the literature are unsuitable to our problem because the Hamiltonian is not regular and we are dealing with vector controls. Consequently, we adopt some ad hoc techniques to develop some useful junction conditions. Note that the states are continuous.

From Eq. (33), it is clear that all of the costates are continuous except possibly for $\lambda_r$ and $\lambda_V$, which may have a discontinuity at a junction point given by

$$\lambda_r(\tau^+) = \lambda_r(\tau^-) + \eta\beta N Q \tag{38}$$

$$\lambda_V(\tau^+) = \lambda_V(\tau^-) - \eta M Q/V \tag{39}$$

Thus, from the continuity of $\lambda_\psi$ and $\lambda_\gamma$ we have the following result.

*Theorem 1:* The bank angle is continuous over the entire trajectory unless the bank angle is singular.

In addition, from Eqs. (35), (37), and (38), we have the following lemma.

*Lemma 1:* Whenever $\lambda_r$ and $\lambda_V$ jump, they do so in an opposing manner,

$$\lambda_r(\tau^+) \geq \lambda_r(\tau^-) \tag{40a}$$

$$\lambda_V(\tau^+) \leq \lambda_V(\tau^-) \tag{40b}$$

*Theorem 2:* It is possible to have an nontangential entry only when all costate variables are continuous at an entry point. If the costates jump, then the entry is tangential.

*Proof:* The condition of Eq. (17) just before the entry point yields

$$T(\tau^-)\cos\alpha(\tau^-) - D(\tau^-) - m\sin\gamma(g + \beta N V^2/M) \geq 0 \tag{41}$$

where strict inequality holds for nontangential entry and equality holds for tangential entry. Right after the entry point, we have

$$T(\tau^+)\cos\alpha(\tau^+) - D(\tau^+) - m\sin\gamma(g + \beta N V^2/M) = 0 \quad (42)$$

Subtracting Eq. (41) from Eq. (42) gives

$$\{T(\tau^-)\cos\alpha(\tau^-) - D(\tau^-)\} - \{T(\tau^+)\cos\alpha(\tau^+) - D(\tau^+)\} \geq 0 \quad (43)$$

From the continuity of Hamiltonian and Eq. (33) and $\mathrm{d}h/\mathrm{d}t = 0$ on the boundary arcs, it is straightforward to show that

$$\lambda(\tau^-)^T[\dot{x}(\tau^-) - \dot{x}(\tau^+)] = 0 \quad (44a)$$

$$\lambda(\tau^+)^T[\dot{x}(\tau^-) - \dot{x}(\tau^+)] = 0 \quad (44b)$$

From Eqs. (44a) and (44b) and continuity of the bank angle and state variables, we obtain

$$\frac{\lambda_V(\tau^-)}{m}[\{T(\tau^-)\cos\alpha(\tau^-) - D(\tau^-)\} - \{T(\tau^+)\cos\alpha(\tau^+) - D(\tau^+)\}] + \left\{\frac{\sin\delta}{mV\cos\gamma}\lambda_\psi(\tau^-) + \frac{\cos\delta}{mV}\lambda_\gamma(\tau^-)\right\} \times[\{T(\tau^-)\sin\alpha(\tau^-) + L(\tau^-)\} - \{T(\tau^+)\sin\alpha(\tau^+) + L(\tau^+)\}] + \frac{\lambda_m(\tau^-)}{v_e}\{T(\tau^+) - T(\tau^-)\} = 0 \quad (45a)$$

and

$$\frac{\lambda_V(\tau^+)}{m}[\{T(\tau^-)\cos\alpha(\tau^-) - D(\tau^-)\} - \{T(\tau^+)\cos\alpha(\tau^+) - D(\tau^+)\}] + \left\{\frac{\sin\delta}{mV\cos\gamma}\lambda_\psi(\tau^+) + \frac{\cos\delta}{mV}\lambda_\gamma(\tau^+)\right\} \times[\{T(\tau^-)\sin\alpha(\tau^-) + L(\tau^-)\} - \{T(\tau^+)\sin\alpha(\tau^+) + L(\tau^+)\}] + \frac{\lambda_m(\tau^+)}{v_e}\{T(\tau^+) - T(\tau^-)\} = 0 \quad (45b)$$

Because $\lambda_\psi$, $\lambda_\gamma$, and $\lambda_m$ are continuous at an entry point, subtracting Eq. (45b) from Eq. (45a) yields

$$(\{\lambda_V(\tau^-) - \lambda_V(\tau^+)\}/m)[\{T(\tau^-)\cos\alpha(\tau^-) - D(\tau^-)\} - \{T(\tau^+)\cos\alpha(\tau^+) - D(\tau^+)\}] = 0 \quad (46)$$

With Eq. (39), Eq. (46) becomes

$$[\eta(\tau)/m](MQ/V)[\{T(\tau^-)\cos\alpha(\tau^-) - D(\tau^-)\} - \{T(\tau^+)\cos\alpha(\tau^+) - D(\tau^+)\}] = 0 \quad (47)$$

If $\eta(\tau)$ is nonzero at an entry point, to satisfy Eq. (47), we have

$$\text{if}\quad \eta(\tau) > 0, \quad\text{then}\quad T(\tau^-)\cos\alpha(\tau^-) - D(\tau^-) = T(\tau^+)\cos\alpha(\tau^+) - D(\tau^+) \quad (48)$$

that is, a tangential entry. On the other hand, if Eq. (43) holds with strict inequality, the entry is nontangential entry, and from Eq. (47), $\eta(\tau) = 0$. □

*Theorem 3:* When all controls are continuous (at an entry point), the costates are also continuous.

*Proof:* Let us suppose that

$$T(\tau^-) = T(\tau^+), \qquad \alpha(\tau^-) = \alpha(\tau^+) \quad (49)$$

From Eq. (25a), we have

$$\frac{L_\alpha + T\cos\alpha}{V\cos\delta} = \frac{\lambda_V(\tau^-)}{\lambda_\gamma}(T\sin\alpha + D_\alpha) \quad (50a)$$

$$\frac{L_\alpha + T\cos\alpha}{V\cos\delta} = \frac{\lambda_V(\tau^+)}{\lambda_\gamma}(T\sin\alpha + D_\alpha) \quad (50b)$$

Subtracting Eq. (50b) from Eq. (50a) yields

$$(\{\lambda_V(\tau^-) - \lambda_V(\tau^+)\}/\lambda_\gamma)(T\sin\alpha + D_\alpha) = 0 \quad (51)$$

With Eq. (39), Eq. (51) becomes

$$[\eta(\tau)/\lambda_\gamma](MQ/V)(T\sin\alpha + D_\alpha) = 0 \quad (52)$$

With assumption of Eq. (9c), we obtain $\eta(\tau) = 0$ from Eq. (52). □

*Theorem 4:* If the thrust switches across a junction point, the angle of attack must jump for a tangential entry.

*Proof:* From Eq. (48), it is clear that if the angle of attack is continuous then thrust must be continuous, but the converse is not necessarily true. Hence, if thrust is discontinuous, the angle of attack must jump to satisfy Eq. (48). □

*Theorem 5:* Theorems 2–4 hold at an exit point.

*Proof:* From the continuity of the Hamiltonian at an exit point and the necessary conditions of costate variables at an exit point and $\mathrm{d}h/\mathrm{d}t = 0$ on the boundary arcs, we obtain

$$\lambda(\tau_2^-)^T\left[\dot{x}(\tau_2^-) - \dot{x}(\tau_2^+)\right] = 0 \quad (53a)$$

$$\lambda(\tau_2^+)^T\left[\dot{x}(\tau_2^-) - \dot{x}(\tau_2^+)\right] = 0 \quad (53b)$$

where $\tau_2$ is the exit point. Using the preceding equation and similar logic used for the entry point, we obtain the same results for the exit point as well. □

Next, from Eqs. (17) and (37), we have

$$\eta[\{T(\tau^-)\cos\alpha(\tau^-) - D(\tau^-)\} - \{T(\tau^+)\cos\alpha(\tau^+) - D(\tau^+)\}] = 0 \quad (54)$$

This equation can also be obtained from the continuity of the Hamiltonian, that is, Eq. (47). An appreciation of these equations can be obtained by considering the special case of the aeroglide maneuver. Because $T(\tau^-) = T(\tau^+) = 0$, Eqs. (43) and (54) simplify to

$$\eta[D(\tau^+) - D(\tau^-)] = 0 \quad (55a)$$

$$[D(\tau^+) - D(\tau^-)] \geq 0 \quad (55b)$$

From the quadratic drag polar, these equations simplify to

$$\eta(C_L^+ + C_L^-)(C_L^+ - C_L^-) = 0 \quad (56a)$$

$$(C_L^+ + C_L^-)(C_L^+ - C_L^-) \geq 0 \quad (56b)$$

If $C_L^+ + C_L^- > 0$, we have

$$\eta(C_L^+ - C_L^-) = 0 \quad (57a)$$

$$(C_L^+ - C_L^-) \geq 0 \quad (57b)$$

Deferring a discussion of the ramifications of $C_L^+ + C_L^- = 0$, it appears that for the aeroglide maneuver, that is, $T = 0$, a nontangential entry cannot be ruled out because the continuity of the Hamiltonian allows an upward jump, that is, $C_L^+ > C_L^-$, in the lift coefficient with continuous costates, that is, $\eta = 0$. Hence, an additional set of jump conditions are necessary to determine the possibility of a nontangential entry. This is obtained from Eq. (15b) as follows. We first note that

$$\left.\frac{\partial H}{\partial\alpha}\right|_{\tau^-} = 0 = \left.\frac{\partial H}{\partial\alpha}\right|_{\tau^+} \quad (58)$$

Hence, from Eq. (15b), we have

$$\lambda_V^-(T\sin\alpha + D_\alpha)^- = \left(\frac{L_\alpha + T\cos\alpha}{V}\right)^-\left(\frac{\lambda_\psi\sin\delta}{\cos\gamma} + \lambda_\gamma\cos\delta\right) \quad (59a)$$

$$\lambda_V^+(T\sin\alpha + D_\alpha)^+ = \left(\frac{L_\alpha + T\cos\alpha}{V}\right)^+\left(\frac{\lambda_\psi\sin\delta}{\cos\gamma} + \lambda_\gamma\cos\delta\right) \quad (59b)$$

Subtracting these two equations, we arrive at

$$\lambda_V^+ \left( \frac{T \sin\alpha + D_\alpha}{L_\alpha + T \cos\alpha} \right)^+ = \lambda_V^- \left( \frac{T \sin\alpha + D_\alpha}{L_\alpha + T \cos\alpha} \right)^- \tag{60}$$

Combining Eq. (60) with Eq. (39), we get

$$\lambda_V^- \left[ \left( \frac{T \sin\alpha + D_\alpha}{L_\alpha + T \cos\alpha} \right)^+ - \left( \frac{T \sin\alpha + D_\alpha}{L_\alpha + T \cos\alpha} \right)^- \right]$$

$$= \frac{\eta M Q}{V} \left( \frac{T \sin\alpha + D_\alpha}{L_\alpha + T \cos\alpha} \right)^+ \tag{61}$$

Using Eqs. (9) and (21), we have

$$\left( \frac{T \sin\alpha + D_\alpha}{L_\alpha + T \cos\alpha} \right)^- \geq \left( \frac{T \sin\alpha + D_\alpha}{L_\alpha + T \cos\alpha} \right)^+ \tag{62}$$

Once again consider the special case of the aeroglide maneuver. Because $T = 0$, Eq. (62) simplifies to

$$\left( \frac{\partial C_D / \partial\alpha}{\partial C_L / \partial\alpha} \right)^- \geq \left( \frac{\partial C_D / \partial\alpha}{\partial C_L / \partial\alpha} \right)^+ \tag{63}$$

Combining this with the drag polar, we get

$$\left( \frac{\partial C_D}{\partial C_L} \right)^- \geq \left( \frac{\partial C_D}{\partial C_L} \right)^+ \tag{64}$$

Thus,

$$(C_L^- - C_L^+) \geq 0 \tag{65}$$

From Eqs. (57b) and (65), we get

$$C_L^- = C_L^+ \tag{66}$$

Thus, we have the following theorem.

*Theorem 6:* For the aeroglide maneuver, all controls ($C_L$ and $\delta$) are continuous at junction points.

From Proposition 1 and Theorem 6, we also obtain the following theorem.

*Theorem 7:* For the aeroglide maneuver, the costate variables are continuous and are characterized by a continuity in the aerodynamic controls.

For a more general maneuver, one with a thrust switching structure, the jump conditions are given by Eqs. (43), (54), and (62). For the aeroglide maneuver, $T = 0$, we first note that $\partial^2 H / \partial \delta^2$ is given by

$$\frac{\partial^2 H}{\partial \delta^2} = -\frac{L}{mV} \left( \lambda_\psi \frac{\sin\delta}{\cos\gamma} + \lambda_\gamma \cos\delta \right) \tag{67}$$

If, in addition, $C_L(\tau) = 0$, then $\partial^2 H / \partial \delta^2 = 0$ at $t = \tau$. Thus, the bank angle is singular at the junction point. It can also be shown that if the Hamiltonian is regular, and if $t = \tau$ is a touch point, then $\partial \dot{h} / \partial \boldsymbol{u} = \boldsymbol{0}$ (Refs. 12 and 13). From Eq. (17), it is clear that (for the aeroglide maneuver)

$$\frac{\partial \dot{h}}{\partial C_L} = 0 \Rightarrow C_L = 0 \tag{68}$$

Thus, a touch point is closely related to a singular point.

## VI. Conclusions

The $D$ form of the Lagrangian was utilized to derive junction conditions for the heating-rate limited optimal synergetic maneuver. In general, jumps in the angle attack and thrust are possible while the bank angle is continuous. These jump conditions may be utilized to include or exclude the possibility of certain switching options. If the atmospheric pass of the synergetic maneuver is further constrained to be performed without any thrusting (that is, the aeroglide maneuver), the lift coefficient—and hence the angle of attack—is continuous. If the lift coefficient vanishes at the junction point, the boundary arc may degenerate to a touch point, rendering the bank angle singular of infinite order. A numerical code that incorporates the conditions derived in this paper may provide additional insight on the true structure of the optimal synergetic maneuver.

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